The Time Ontology of Allen's Interval Algebra

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Abstract

Allen's interval algebra is a set of thirteen jointly exhaustive and pairwise disjoint binary relations representing temporal relationships between pairs of timeintervals. Despite widespread use, there is still the question of which time ontology actually underlies Allen's algebra. Early work specified a first-order ontology that can interpret Allen's interval algebra; in this paper, we identify the first-order ontology that is logically synonymous with Allen's interval algebra, so that there is a one-to-one correspondence between models of the ontology and solutions to temporal constraints that are specified using the temporal relations. We further prove a representation theorem for the ontology, thus characterizing its models up to isomorphism.

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1 Introduction

Temporal reasoning has long been studied in artificial intelligence, particularly since the seminal work of Allen, in which time is represented using binary relations over intervals. The composition of these relations leads to an algebra, which have been widely used for constraint satisfaction problems. Today, virtually every presentation of a time ontology includes at least the diagram of the temporal relations in Allen's algebra and the composition table for the relations. This work was later extended by Hayes and Allen, who proposed a first-order ontology corresponding to Allen's algebra; additional extensions were proposed by Ladkin and Maddox.

Given this long story, it might be surprising that there is anything left to say; yet a closer inspection of the ontologies involved leads to some interesting observations. First, nobody has shown which ontology of time intervals is equivalent to Allen's Interval Algebra; previous work has only shown that a first-order axiomatization of the algebra is interpreted by a particular axiomatization of an ontology of time intervals.

Second, there has been no characterization up to isomorphism of the models of the first-order axiomatization of Allen's Interval Algebra. The closest work along these lines has been a discussion of the models of time interval ontologies, but this is far short of a full characterization. The models are often informally specified; the more formal specifications of the models refer to intervals of integers or rational numbers, rather than an explicit formal specification in the signature of the ontologies. Furthermore, there have been no proofs of representation theorems for these classes of models that do not refer to an underlying set of points. Finally, the relationship between different ontologies of time intervals has not been

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fully explicated. The relationship between these other interval ontologies and Allen's Interval Algebra is thus also not clear.

In this paper we investigate Allen's Interval Algebra based on the metalogical relationship between the first-order theory of the composition table and the different axiomatizations of time ontologies of intervals. After reviewing the basic axiomatizations of the time ontologies in Section 2, we discuss the relationship between the theory of the interval algebra T_{allen} and Hayes' axiomatization ($T_{interval\_meeting}$) in Section 3. After showing that $T_{interval\_meeting}$ cannot be interpreted by T_{allen}, we propose a new ontology $T_{bounded\_meeting}$, which is weaker than $T_{interval\_meeting}$. Our key result is that a nonconservative extension of the interval algebra, which we call T^*_{allen}, is logically synonymous with the $T_{bounded\_meeting}$, meaning $T_{bounded\_meeting}$ and T^*_{allen} axiomatize the same class of structures. In other words,the two theories are semantically equivalent, and only differ in signature (i.e., the non-logical symbols). Finally, in Section 4, we present a characterization of models of $T_{bounded\_meeting}$ up to isomorphism, and explain how such a characterization can be used in characterizing algebraic properties of models of T_{allen}.

2 Preliminaries

2.1 Allen's Interval Algebra

Allen's introduction of thirteen relations over temporal intervals [2] laid the foundations for qualitative temporal reasoning and representation. The interval relations are $meets$, $before$, $starts$, $ends$, $overlaps$, $during$, their inverses met_by, $after$, $started\_by$, $ended\_by$, $overlapped\_by$, $contains$, and equality. These relations are pairwise disjoint and exhaustive (that is, any two time intervals must be related by one of these relations). The notion of an algebra over these relations arises from considering the intersection, union, and composition of a pair of temporal relations. This leads to the composition table CT, which is a 13×13 matrix such that for each ordered pair of interval relations R_i,R_j, the cell $CT(R_i, R_j)$ indicates the possible temporal relations between two intervals a and c assuming that $R_i(a, b)$ and $R_j(b, c)$ holds. For example, $CT(starts, overlaps) = \{before, overlaps, meets\}$, meaning that if $starts(a, b)$ and $overlaps(b, c)$, then the interval a is before, overlaps, or meets the interval c.

2.2 Ontologies for Time Intervals

Although the application of Allen's interval algebra was widespread in the specification and solution of temporal constraint satisfaction problems, it was several years before people considered its relationship to the time ontologies being developed within the knowledge representation community. In this section, we review the primary time ontologies that are relevant to the axiomatization of the interval algebra in first-order logic.

Some of the earliest time ontologies [12] treated timepoints as the primitive entities in the domain. However, the entities for the interval algebra are time *intervals* – points do not exist. The first proposal for the axiomatization of an ontology[1] of time intervals as related to Allen's interval algebra was the work of Hayes [7], [1], in which there is one primitive binary relation $meets$ over intervals. This axiomatization, which we will refer to as $T_{interval\_meeting}$ is shown in Figure 1.

[1] A theory is set of first-order sentences closed under logical entailment. In this paper, we use the terms ontology and theory interchangably.

$$(\forall i,j,k,m)\ meets(i,k) \wedge meets(j,k) \wedge meets(i,m) \supset meets(j,m) \quad (1)$$
$$(\forall i)(\exists j,k)\ meets(j,i) \wedge meets(i,k) \quad (2)$$
$$(\forall i,j,k,l)\ (meets(i,j) \wedge meets(k,l)) \supset$$
$$(meets(i,l) \vee ((\exists n)\ ((meets(i,n) \wedge meets(n,l)) \vee (meets(k,n) \wedge meets(n,j))))) \quad (3)$$
$$(\forall i,j)\ meets(i,j) \supset \neg meets(j,i) \quad (4)$$
$$(\forall i,j,k,m)\ meets(i,j) \wedge meets(j,k) \wedge meets(k,m) \supset (\exists n)\ meets(i,n) \wedge meets(n,m) \quad (5)$$

Figure 1 The axioms of $T_{interval\_meeting}$.

$$(\forall i,j)\ before(i,j) \equiv (\exists k)\ meets(i,k) \wedge meets(k,j) \quad (6)$$
$$(\forall i,j)\ starts(i,j) \equiv (\exists k,m,n)\ meets(k,i) \wedge meets(i,m)$$
$$\wedge meets(m,n) \wedge meets(k,j) \wedge meets(j,n) \quad (7)$$
$$(\forall i,j)\ ends(i,j) \equiv (\exists k,m,n)\ meets(k,m) \wedge meets(m,i)$$
$$\wedge meets(i,n) \wedge meets(k,j) \wedge meets(j,n) \quad (8)$$
$$(\forall i,j)\ overlaps(i,j) \equiv (\exists k,m,n,o,p)\ meets(k,m) \wedge meets(m,n)$$
$$\wedge meets(n,o) \wedge meets(o,p) \wedge meets(m,j) \wedge meets(j,p) \wedge meets(k,i) \wedge meets(i,o) \quad (9)$$
$$(\forall i,j)\ during(i,j) \equiv (\exists k,m,n.o)\ meets(k,m) \wedge meets(m,i)$$
$$\wedge meets(i,n) \wedge meets(n,o) \wedge meets(k,j) \wedge meets(j,o) \quad (10)$$

Figure 2 $T_{interval\_rel}$: the definitions for Allen's Interval Algebra relations.

By Axiom (1) if two intervals meet a common interval, then the sets of intervals that each meet is equivalent to each other. For each time interval, Axiom (2) guarantees the existence of an interval that it *meets*, and an interval that is met by it. Since this leads to infinite models, we will refer to this as the Infinity Axiom. Axiom (3) captures the intuition that the *meets* relation leads to an ordering over time intervals. By Axiom (4), the *meets* relation is asymmetric. Axiom (5) is often referred to as the Sum Axiom, since it entails the existence of an interval that is formed by the union of two intervals that meet.

The axiomatization of $T_{interval\_meeting}$ is sufficient to enable the definition of the relations in Allen's interval algebra (see Figure 2). $T_{interval\_meeting} \cup T_{interval\_rel}$ is therefore a definitional extension of $T_{interval\_meeting}$. This extension will play a key role in determining the relationship between $T_{interval\_meeting}$ and the interval algebra.

Hayes describes the models of $T_{interval\_meeting}$ and its extensions in terms of the set of intervals on $\mathbb{Q}$ (rational numbers) and $\mathbb{Z}$ (integers). For example, he describes one model which interprets intervals as open connected subsets of $\mathbb{Q}$, such that (a,b) meets (c,d) when $a = c$ and the intersection of two meeting intervals is empty. Alternative models exist, which interpret intervals as closed connected subsets. Analogous models are also described as sets of closed intervals on $\mathbb{Z}$.

This treatment is inadequate for several reasons. Strictly speaking, a structure on the sets of intervals on $\mathbb{Q}$ is not a model of $T_{interval\_meeting}$ because it does not have the same signature; at best, this is saying that models of $T_{interval\_meeting}$ can be interpreted in such a structure. Yet even this falls short because it does not determine whether *all* models can be interpreted in this way, or whether there exist other models which must be constructed in a different fashion (i.e. do there exist models of the ontology that are not isomorphic to intervals over $\mathbb{Q}$ or $\mathbb{Z}$?). Furthermore, the description is informal, without a formal proof of equivalence; there is no explicit definition or characterization of the models of $T_{interval\_meeting}$ in and of themselves.

Several extensions to $T_{interval\_meeting}$ have been proposed. The work of Ladkin ([9], in particular explored an extension which is categorical. Ladkin provides a more formal characterization for the models of his axiomatization which also specifies intervals as pairs of points in an underlying linear ordering. As with Hayes, this is essentially a representation theorem for models of the ontology, rather than a direct characterization of the models themselves.

3 Relationship between Allen's Interval Algebra and Time Ontologies

Although [7] stated that Allen's Interval Algebra can be derived from the ontology $T_{interval\_meeting}$, the two approaches display different properties. In particular, all models of $T_{interval\_meeting}$ are infinite, whereas Allen's Interval Algebra allows finite models. These differences raise the question of which ontology actually underlies the interval algebra.

We begin this section by describing the logical theory that captures the composition table for Allen's Interval Algebra, and then search for the time ontology that is equivalent to it.

3.1 First-Order Theory of Allen's Interval Algebra

To specify the first-order theory T_{allen} of Allen's Interval Algebra, we follow [3] and we assume that for each cell in the composition table, we have a first-order sentence of the form

$$R_i(x,y) \wedge R_j(y,z) \supset T_1(x,z) \vee ... \vee T_n(x,z)$$

where $CT(R_i, R_j) = \{T_1, ..., T_n\}$. For example, the following sentence is the axiom in which corresponds with $CT(meets, ends)$:

$$meets(x,y) \wedge ends(y,z) \supset (overlaps(x,z) \vee during(x,z) \vee starts(x,z)).$$

Since the composition table consists of 13×13 cells, T_{allen} must contain 169 axioms corresponding with the table. We will denote this set of axioms as $T_{allen\_compose}$.

In addition to these axioms, we assume that for each interval algebra relation R_1, T_{allen} contains a sentence of the following form stating that the relations are pairwise disjoint (PD):

$$R_1(x,y) \supset \neg(R_2(x,y) \vee ... \vee R_{13}(x,y))$$

where $R_2, ..., R_{13}$ are interval algebra relations other than R_1. The following sentence, for example, is the PD axiom corresponding with *meets*:

$$\begin{aligned} meets(x,y) \supset \neg[&before(x,y) \vee starts(x,y) \vee ends(x,y) \vee overlaps(x,y) \vee during(x,y) \\ &\vee met\_by(x,y) \vee after(x,y) \vee started\_by(x,y) \vee ended\_by(x,y) \\ &\vee overlapped\_by(x,y) \vee contains(x,y) \vee (x=y)]. \end{aligned}$$

As there are 13 interval algebra relations, T_{allen} contains 13 PD axioms; we will denote the disjointness axioms by $T_{allen\_disjoint}$.

Finally, T_{allen} contains an axiom that specifies that the interval algebra relations are jointly exhaustive:

$$meets(x, y) \lor before(x, y) \lor starts(x, y) \lor ends(x, y) \lor ends(x, y) \lor overlaps(x, y)$$
$$\lor during(x, y) \lor \lor met\_by(x, y) \lor after(x, y) \land started\_by(x, y)$$
$$\lor ended\_by(x, y) \land overlapped\_by(x, y) \lor contains(x, y) \lor (x = y).$$

We will refer to this axiom as $T_{exhaustive}$.

All other sentences in T_{allen} are those which are entailed by the $169 + 13 + 1$ above-mentioned axioms. Thus,

$$T_{allen} = T_{allen\_compose} \cup T_{allen\_disjoint} \cup T_{exhaustive}$$

Models of T_{allen} are equivalent to solutions of temporal constraints that are expressed using the interval relations, but the question of a characterization of models of T_{allen} remains unresolved. In the following subsections, we identify the time ontology that is equivalent to T_{allen}, and in the latter part of the paper, we characterize the models of this time ontology up to isomorphism.

3.2 T_{allen} and $T_{interval\_meeting}$

Hayes and Allen ([1]) state that the interval algebra composition table can be derived from the ontology of time intervals. More precisely, the first-order theory of the interval algebra composition table can be entailed from a definitional extension of $T_{interval\_meeting}$:

▶ **Definition 1** (adopted from [8]). Let T be a first-order theory and Π be a set containing sentences of the following form [2]

$$R(x_1, ..., x_n) \equiv \Phi(x_1, ..., x_n)$$

where R is a predicate which is not in $\Sigma(T)$ and Φ is a formula in $\mathcal{L}(T)$ in which at most variables $x_1, ..., x_n$ occur free. $T \cup \Pi$ is called a *definitional extension* of T.

▶ **Theorem 2.** *T_{allen} is entailed by $T_{interval\_meeting} \cup T_{interval\_rel}$.*

Proof. Using the automated theorem prover Prover9 [10], we have shown[3] that $T_{interval\_meeting} \cup T_{interval\_rel}$ entails for each axiom Φ of T_{allen},

$$T_{interval\_meeting} \cup T_{interval\_rel} \models T_{allen}$$

◀

This Theorem is equivalent to saying that T_{allen} has an interpretation in $T_{interval\_meeting}$. A theory T_1 has a relative interpretation [5] in another theory T_2 if every sentence in T_1 can be translated into a sentence in T_2. In other words, for all sentences $\Phi \in \mathcal{L}(T_1)$, if T_1 entails

[2] For a theory T, $\Sigma(T)$ denotes the *signature* of T, i.e., the set of non-logical symbols used in sentences of T; $\mathcal{L}(T)$ denotes the *language* of T, i.e., the set of all first-order formulae generated by symbols in $\Sigma(T)$; $Mod(T)$ denotes the class of all models of T.

[3] The input files and proofs can be found at `colore.oor.net/allen_interval_algebra/mappings/theorems/intervalmeeting2allen/`

Φ, then T_2 entails a translation of Φ into the language of T_2. The work of [6] shows that if a definitional extension of T_2 entails T_1, translations for sentences of T_1 is obtained based on the formulas which define predicates of T_1 in the definitional extension. For instance, a translation of Axiom (5) of $T_{interval\_meeting}$ into the language of T_{allen} can be obtained by replacing formulas with $before$ literals (whose definition can be found in Figure 2). The result is the following sentence, which provably is a sentence in T_{allen}:

$$(\forall i, j, m) before(i, j) \wedge meets(j, m) \supset before(i, m)$$

Theories T_1 and T_2 are mutually interpretable iff they are relatively interpretable in each other. In our case, $T_{interval\_meeting}$ is not relatively interpretable in T_{allen}, since T_{allen} cannot entail all axioms of $T_{interval\_meeting}$:

▶ Proposition 1.

$$T_{allen} \not\models (\forall i)(\exists j)\, meets(i, j)$$

Proof. The model generated by Mace4 can be found at
`colore.oor.net/allen_interval_algebra/mappings/theorems/allen2boundedmeeting/finite.model/`. ◀

By Proposition 1, T_{allen} does not interpret $T_{interval\_meeting}$ because it allows finite models, whereas all models of $T_{interval\_meeting}$ are infinite. To achieve mutual interpretability, we need to weaken $T_{interval\_meeting}$, However, simply removing the infinity axiom from $T_{interval\_meeting}$ doesn't work:

▶ Proposition 2. Let T_{finite} be the set of all axioms in $T_{interval\_meeting}$ except Axiom (2).
$T_{finite} \cup T_{interval\_rel} \models T_{allen\_compose} \cup T_{allen\_disjoint}$
$T_{finite} \cup T_{interval\_rel} \not\models T_{exhaustive}$

Proof. In the proofs of Theorem 2, Axiom (2) is not used to entail any sentence in $T_{allen\_compose}$ or $T_{allen\_disjoint}$.

A model of $T_{finite} \cup T_{interval\_rel}$ that falsifies $T_{exhaustive}$ can be found at
`colore.oor.net/allen_interval_algebra/mappings/theorems/allen2boundedmeeting/exhaustive.model`. ◀

We therefore need a theory that is stronger than T_{finite} but weaker than $T_{interval\_meeting}$. The natural question is therefore: what is the theory in the $\mathbb{H}^{interval\_meeting}$ Hierarchy that is entailed by T_{allen}?

3.3 Bounded Meeting

In this section, we search for a theory that is weaker than $T_{interval\_meeting}$, yet which is still able to interpret T_{allen}. We begin by taking a closer look at the role that the Infinity Axiom plays in the proofs for Theorem 2. This axiom guarantees that each interval is bounded by an earlier and later interval. If we also look at the definitions of the interval relations *starts*, *ends*, *overlaps*, and *during*, we see that each definition entails the existence of two intervals – one that is earlier than the others and one that is later than the others.

Inspired by this observation, we propose the definition of a new relation, *prec*, which specifies an ordering over intervals. The new axioms guarantee the existence of lower and upper bounds for each pair of intervals with respect to this ordering.

$$(\forall i,j)\ meets(i,j) \supset timeinterval(i) \wedge timeinterval(j) \tag{11}$$
$$(\forall i,j,k,m)\ meets(i,k) \wedge meets(j,k) \wedge meets(i,m) \supset meets(j,m) \tag{12}$$
$$(\forall i,j,k,l)\ (meets(i,j) \wedge meets(k,l)) \supset$$
$$(meets(i,l) \vee ((\exists n)\ ((meets(i,n) \wedge meets(n,l)) \vee (meets(k,n) \wedge meets(n,j))))) \tag{13}$$
$$(\forall i,j)\ meets(i,j) \supset \neg meets(j,i) \tag{14}$$
$$(\forall i,j,k,m)\ meets(i,j) \wedge meets(j,k) \wedge meets(k,m) \supset$$
$$(\exists n)\ meets(i,n) \wedge meets(n,m) \tag{15}$$
$$(\forall x,y)(\exists z)\ prec(x,z) \wedge prec(y,z) \tag{16}$$
$$(\forall x,y)(\exists z)\ prec(z,x) \wedge prec(z,y) \tag{17}$$
$$(\forall x,y)\ prec(x,y) \equiv (meets(x,y) \vee ((\exists z)\ meets(x,z) \wedge meets(z,y)) \vee (x=y)) \tag{18}$$

Figure 3 The axioms of $T_{bounded\_meeting}$.

▶ Proposition 3.

$$T_{interval\_meeting} \models (\forall x,y)(\exists z)\ prec(x,z) \wedge prec(y,z)$$

$$T_{interval\_meeting} \models (\forall x,y)(\exists z)\ prec(z,x) \wedge prec(z,y)$$

Proof. The proof generated by Prover9 can be found at
`colore.oor.net/allen_interval_algebra/theorems/interval-bounded/`. ◀

Thus, $T_{bounded\_meeting}$ is entailed by $T_{interval\_meeting}$, and it entails T_{finite}.

3.4 T_{allen} and $T_{bounded\_meeting}$

Although $T_{bounded\_meeting}$ is weaker than $T_{interval\_meeting}$, it is strong enough to relatively interpret T_{allen}:

▶ **Theorem 3.** *T_{allen} is entailed by $T_{bounded\_meeting} \cup T_{interval\_rel}$.*

Proof. Using Prover9, we have shown[4] that

$$T_{bounded\_meeting} \cup T_{interval\_rel} \models \Phi$$

for each axiom Φ of T_{allen}. ◀

Unlike $T_{interval\_meeting}$, the theory $T_{bounded\_meeting}$ allows finite models, but is it weak enough to be interpreted by T_{allen}?

▶ Proposition 4.

$$T_{allen} \not\models (\forall i,j)(\exists k)(meets(i,k) \vee before(i,k) \vee (i=k)) \wedge (meets(j,k) \vee before(j,k) \vee (k=j))$$

[4] The input files and proofs can be found at
`colore.oor.net/allen_interval_algebra/mappings/theorems/boundedmeeting2allen/`

$$(\forall x, y)\ before(x, y) \supset (\exists z)\ meets(x, z) \wedge meets(z, y) \tag{19}$$
$$(\forall x, y)\ overlaps(x, y) \supset (\exists z)\ ends(z, x) \wedge starts(z, y) \tag{20}$$
$$(\forall x, y)\ during(x, y) \supset (\exists z)\ ends(x, z) \wedge starts(z, y) \tag{21}$$
$$(\forall x, y)\ starts(x, y) \supset (\exists z)\ meets(z, x) \wedge meets(z, y) \tag{22}$$
$$(\forall x, y)\ ends(x, y) \supset (\exists z)\ meets(x, z) \wedge meets(y, z) \tag{23}$$
$$(\forall x, z)\ starts(x, z) \supset (\exists y)\ before(x, y) \wedge meets(z, y) \tag{24}$$
$$(\forall x, z)\ ends(x, z) \supset (\exists y)\ before(y, x) \wedge meets(y, z) \tag{25}$$

Figure 4 T_{allen_exist}: Additional axioms to extend T_{allen}.

Proof. The model generated by Mace4 that falsifies the sentence can be found at `colore.oor.net/allen_interval_algebra/mappings/theorems/allen2boundedmeeting/bounded.model/`. ◀

Thus, T_{allen} cannot interpret $T_{bounded\_meeting}$ either, yet a cursory glance at the definitions of the temporal relations in $T_{interval\_rel}$ seems to indicate that it should. In the preceding section, we used the consistency-based definition [3] to axiomatize Allen's Interval Algebra. An alternative approach to the axiomatization of the composition table is known as the extensional definition approach [3] – the composition of R_1 with R_2 is the set of ordered pairs $\langle \mathbf{x}, \mathbf{y} \rangle$ such that for some $\mathbf{z}$, we have $\langle \mathbf{x}, \mathbf{z} \rangle \in R_1$ and $\langle \mathbf{z}, \mathbf{y} \rangle \in R_2$. For example, the extensional definition axiom for the cell $CT(meets, meets)$ is

$$(\forall x, y)\ before(x, y) \equiv (\exists z)\ meets(x, z) \wedge meets(z, y)$$

If we look closely, we notice that there are sentences from the extensional definition of the composition table that are entailed by the definitions of the relations in Figure 2. In particular, each sentence in Figure 4 is entailed by $T_{interval\_meeting} \cup T_{interval\_rel}$, and each of these sentences corresponds to the converse of an axiom from the consistency-based definition of the composition table. Since we are ultimately interested in understanding the relationship between $T_{interval\_meeting}$ and the composition table, we will extend the theory T_{allen} with the sentences of Figure 4, and refer to the resulting theory as T^*_{allen}.

▶ **Lemma 4.** *$T_{bounded\_meeting} \cup T_{interval\_rel}$ entails T^*_{allen}.*

Proof. Using Prover9, we have shown[5] that

$$T_{bounded\_meeting} \cup T_{interval\_rel} \models \Phi$$

for each axiom Φ of T_{allen_exist} (and hence T^*_{allen}). ◀

▶ **Lemma 5.** *T^*_{allen} entails $T_{bounded\_meeting}$.*

[5] The input files and proofs can be found at `colore.oor.net/allen_interval_algebra/mappings/theorems/boundedmeeting2allen/`

Proof. Using Prover9, we have shown[6] that $T^*_{allen} \models \Phi$ for each axiom Φ of $T_{bounded\_meeting}$. ◀

These two Lemmas, it can be shown that T_{allen} and $T_{bounded\_meeting}$ are *mutually interpretable* [5] in each other. Relative interpretation alone does not guarantee a one-to-one correspondence between models of the theories. When T_1 is interpretable in T_2, we can only show every model of T_2 defines a model of T_1 using the translation definitions between T_1 and T_2. Thus, establishing mutual relative interpretation between $T_{bounded\_meeting}$ and T_{allen} does not help with characterizing all models of T_{allen}.

To study models of T_{allen} based on models of $T_{bounded\_meeting}$, we need a notion stronger than relative interpretation:

▶ **Definition 6** ([8])**.** Two theories T_1 and T_2 are logically synonymous iff they have a common definitional extension.

Considering Definition 6, it is easy to see that T_1 and T_2 are synonymous iff there exist two sets of translation definitions, Δ and Π, such that $T_1 \cup \Pi$ is a definitional extension of T_1, $T_2 \cup \Delta$ is a definitional extension of T_2, and $T_1 \cup \Pi$ and $T_2 \cup \Delta$ are logically equivalent.

When two theories are synonymous, there is a one-to-one correspondence between their models such that the corresponding models can be defined based on each other [11].

▶ **Theorem 7.** *$T_{bounded\_meeting}$ is logically synonymous with T^*_{allen}.*

Proof. By Lemma 4 and Lemma 5, $T_{bounded\_meeting}$ and T^*_{allen} are mutually interpretable.

Using Prover9, we have shown[7] that $T^*_{allen} \models \Phi$ for each axiom Φ of $T_{interval\_rel}$.

Thus, $T_{bounded\_meeting} \cup T_{interval\_rel}$ and T^*_{allen} are logically equivalent. ◀

According to [11], synonymous theories axiomatize the same class of structures. Thus, $T_{bounded\_meeting}$ and T^*_{allen} are semantically equivalent and only differ in signature.

In what sense has this achieved our objective, since we have we have shown that the time ontology $T_{bounded\_meeting}$ is synonymous with an extension of first-order axiomatization of Allen's interval algebra, rather than the original axiomatization? First, the additional axioms in T_{allen_exist} that we need to prove Theorem 7 are all entailed from the definitions of the interval relations; any attempt to use a weaker axiomatization of the composition table would require us to change these defintions. However, these definitions capture the intended semantics of the interval relations, so any weaker definition would lead to unintended models. The underlying ontology of time intervals therefore plays no role in entailment of the axioms in T_{allen_exist}.

All relations in the composition table can be deduced from T^*_{allen} as it is an extension of T_{allen}. In addition, for every entry $CT(R_i, R_j)$ of the composition table and every interval algebra relation $S \notin CT(R_i, R_j)$ we proved a sentence of the form: $R_i(x, y) \wedge R_j(y, z) \supset \neg S(x, z)$

Thus, the additional axioms of T^*_{allen} does not change the composition table, but only eliminate those models of T_{allen} that do not satisfy the axiomatic definitions of the interval relations.

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[6] The input files and proofs can be found at `colore.oor.net/allen_interval_algebra/mappings/theorems/allen2boundedmeeting/`

[7] The input files and proofs can be found at `colore.oor.net/allen_interval_algebra/mappings/theorems/allen2boundedmeeting/`

4 Model-Theoretic Characterization of $T_{bounded\_meeting}$

It is tempting to see the equivalence between T_{allen} and $T_{bounded\_meeting}$ as an intellectual curiosity that does not give us any new insights into the interval algebra. Nevertheless, if we recall that Allen's Interval Algebra is primarily used in constraint satisfaction problems, in which one constructs a satisfying interpretation of a set of expressions in the signature of T_{allen}, then the set of all possible solutions of interval algebra problems, excluding those eliminated by T_{allen}, is equivalent to the set of all possible models of $T_{bounded\_meeting}$. In this section, we provide a characterization of the models[8] of $T_{bounded\_meeting}$ up to isomorphism, by first specifying a class of mathematical structures, and then showing that $T_{bounded\_meeting}$ axiomatizes this class of structures.

4.1 Representation Theorem for Models of $T_{bounded\_meeting}$

Verification is concerned with the relationship between the models of the axiomatization of the ontology and a class of mathematical structures. In particular, we want to characterize the models of an ontology up to isomorphism and determine whether or not these models are equivalent to the intended models of the ontology. In this section, we characterize the models of $T_{bounded\_meeting}$.

Since **meets** is a asymmetric binary relation, we turn to directed graphs for the underlying structures:

▶ **Definition 8.** A directed graph is a pair $\langle V, A\rangle$ such that $A \subseteq V \times V$.

Before stating the representation theorem, we need some notation.

▶ **Definition 9.** Let $\mathcal{M} = (V, A)$ be a directed graph. For each $\mathbf{x} \in V$,

$$N(\mathbf{x}) = \{\mathbf{y} : (\mathbf{x}, \mathbf{y}) \in A\} \qquad N^{-1}(\mathbf{x}) = \{\mathbf{y} : (\mathbf{y}, \mathbf{x}) \in A\}$$

$$N^k(\mathbf{x}) = N(N^{k-1}(\mathbf{x})) \qquad N^{-k}(\mathbf{x}) = N^{-1}(N^{-(k-1)}(\mathbf{x}))$$

$$D^k(\mathbf{x}) = \bigcup_{i=1}^{i=k} N^i(\mathbf{x}) \qquad D^{-k}(\mathbf{x}) = \bigcup_{i=1}^{i=k} N^{-i}(\mathbf{x})$$

▶ **Definition 10.** $\mathfrak{M}^{bounded\_meeting}$ is the following class of structures:
$\mathcal{M} \in \mathfrak{M}^{bounded\_meeting}$ iff $\mathcal{M} = (V, A)$ is a directed graph such that

1. $N(\mathbf{x}) \cap N^2(\mathbf{x}) = \emptyset$ for any $\mathbf{x} \in V$;
2. $N^3(\mathbf{x}) \subseteq N^2(\mathbf{x})$ for any $\mathbf{x} \in V$;
3. for any $\mathbf{x}, \mathbf{y} \in V$,

$$D^2(\mathbf{x}) \cap D^2(\mathbf{y}) \neq \emptyset$$

$$D^{-2}(\mathbf{x}) \cap D^{-2}(\mathbf{y}) \neq \emptyset$$

4. $N(\mathbf{x}) \subseteq N^2(\mathbf{y})$ or $N(\mathbf{y}) \subseteq D^2(\mathbf{x})$, for any $\mathbf{x}, \mathbf{y} \in V$,

▶ **Theorem 11.** *There exists a bijection*
$\varphi : Mod(T_{bounded\_meeting}) \to \mathfrak{M}^{bounded\_meeting}$ *such that*

---

[8] We denote *structures* by calligraphic uppercase letters, e.g. $\mathcal{M}, \mathcal{N}$; elements of a structure by **boldface** font, e.g., $\mathbf{a}, \mathbf{b}$; and the *extension of predicate* R in a structure $\mathcal{M}$ by $\mathbf{R}^{\mathcal{M}}$.

1. $\langle \mathbf{x} \rangle \in \mathbf{timeinterval}$ *iff* $\mathbf{x} \in V^{\varphi(\mathcal{M})}$;
2. $\langle \mathbf{x}, \mathbf{y} \rangle \in \mathbf{meets}^{\mathcal{M}}$ *iff* $(\mathbf{x}, \mathbf{y}) \in A^{\varphi(\mathcal{M})}$.

Proof. (Sketch) Condition 1 in Definition 10 is equivalent to the sentence

$$(\forall x, y, z)\, meets(x, y) \wedge meets(y, z) \supset \neg meets(x, z)$$

and using Prover9[9], we can show that this sentence is logically equivalent to Axiom (12).

Condition 2 in Definition 10 is equivalent to Axiom (15).

Condition 3 in Definition 10 is equivalent to Axioms (16) and (16).

Condition 4 in Definition 10 is equivalent to Axiom (13).

Axiom (14) is equivalent to the property that $\mathcal{M}$ is a directed graph. ◀

We can therefore refer interchangably to the models of $T_{bounded\_meeting}$ and structures in $\mathfrak{M}^{bounded\_meeting}$.

4.2 Representation Theorem for $\mathfrak{M}^{bounded\_meeting}$

Although Theorem 11 characterizes the models of $T_{bounded\_meeting}$, it leaves unresolved the explicit characterization of $\mathfrak{M}^{bounded\_meeting}$, and a deeper understanding of how to construct models of $T_{bounded\_meeting}$. In this section, we characterize the class of directed graphs that satisfy the conditions in Definition 10.

Although structures in $\mathfrak{M}^{bounded\_meeting}$ are directed graphs, it will be easier to construct them from undirected graphs, allowing us to exploit a wider range of existing work in graph theory.

▶ **Definition 12.** An undirected graph is a pair $G = \langle V, E \rangle$ of sets such that $E \subseteq \{V\}^2$.

Directed and undirected graphs are related to each other through the notion of orientation.

▶ **Definition 13.** $G_1 = \langle V, A \rangle$ is an orientation of an undirected graph $G_2 = \langle V, E \rangle$ iff for each $(\mathbf{x}, \mathbf{y}) \in E$, either $(\mathbf{x}, \mathbf{y}) \in A$ or $(\mathbf{y}, \mathbf{x}) \in A$.

If G_1 is a directed graph, then G_2 is the undirected graph for G_1 iff G_1 is an orientation of G_2.

We therefore need to identify the class of graphs that correspond to structures in $\mathfrak{M}^{bounded\_meeting}$.

4.2.1 Twin-free Graphs

We first notice that

$$T_{bounded\_meeting} \not\models (\forall x, y, z, u) meets(x, y) \wedge meets(x, z) \wedge meets(y, u) \wedge meets(z, u) \supset (y = z)$$

Thus there exist models in which there exist multiple intervals that meet and are met by the same intervals.

▶ **Definition 14.** Let $G = (V, E)$ be a graph. Two vertices $\mathbf{x}, \mathbf{y} \in V$ are twins iff for all other vertices $\mathbf{w}$, we have $(\mathbf{x}, \mathbf{w}) \in E$ iff $(\mathbf{y}, \mathbf{w}) \in E$.

G is twin-free iff it contains no twins.

[9] `color.oor.net/allen_interval_algebra/theorems/triangle_free/`

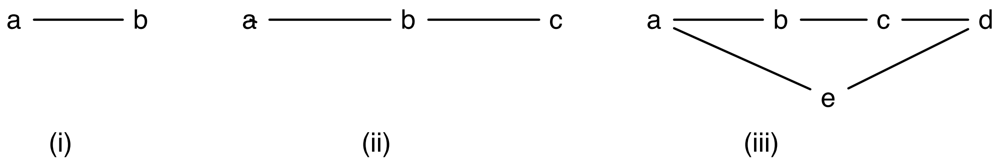

Figure 5 Short models.

▶ **Definition 15.** Suppose $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are graphs such that $V_1 \subset V_2$ and $E_1 \subset E_2$. G_2 is a twinned extension of G_1 iff each $\mathbf{x} \in V_2 \setminus V_1$ is a twin of some $\mathbf{y} \in V_1$.

▶ **Lemma 16.** *If $\mathcal{M} \in \mathfrak{M}^{bounded\_meeting}$, and $\mathcal{N}$ is a twinned extension of $\mathcal{M}$, then $\mathcal{N} \in \mathfrak{M}^{bounded\_meeting}$.*

Proof. If $\mathbf{x}, \mathbf{y} \in V$ are twins, then

$$N^k(\mathbf{x}) = N^k(\mathbf{y}) \qquad N^{-k}(\mathbf{x}) = N^{-k}(\mathbf{y})$$

so that $\mathcal{N}$ satisfies all conditions in Definition 10 and hence $\mathcal{N} \in \mathfrak{M}^{bounded\_meeting}$. ◀

▶ **Lemma 17.** *Each $\mathcal{M} \in \mathfrak{M}^{bounded\_meeting}$ contains a unique twin-free subgraph.*

Proof. If $\mathbf{x}, \mathbf{y} \in V$ are twins, then removing $\mathbf{y}$ does not change $N^k(\mathbf{x})$ or $N^{-k}(\mathbf{x})$, so that the conditions in Definition 10 will be satisfied by the subgraph of $\mathcal{M}$ that is induced by $V \setminus \{\mathbf{y}\}$. ◀

Thus, we can characterize $\mathfrak{M}^{bounded\_meeting}$ by characterizing the twin-free graphs in $\mathfrak{M}^{bounded\_meeting}$.

4.2.2 Building Blocks

We now consider some special graphs that serve as the basic structures from which models of $T_{bounded\_neeting}$ are constructed.

▶ **Definition 18.** A short model is an undirected graph that is isomorphic to either P_2 (path graph with two vertices), P_3 (path graph with three vertices), or C_5 (cyclic graph with five vertices).

The three short models are depicted in Figure 5, and they each correspond to a different set of temporal relations. The orientation of P_2 is the smallest model of $T_{bounded\_meeting}$ with a nontrivial extension of the **meets** relation. The orientation of P_3 is the smallest model of $T_{bounded\_meeting}$ with a nontrivial extension of the **before** relation. The orientation of C_5 is the smallest model of $T_{bounded\_meeting}$ with a nontrivial extension of the **starts** and **ends** relations.

Note that the orientation of a cyclic undirected graph can be a directed acyclic graph.

▶ **Lemma 19.** *If $\mathcal{M}$ is an orientation of a short model G, then $\mathcal{M} \in \mathfrak{M}^{bounded\_meeting}$.*

▶ **Definition 20.** The $\mathbb{H}$ graph is the connected graph on six vertices such that exactly two vertices have degree 3 and the remaining vertices have degree 1.

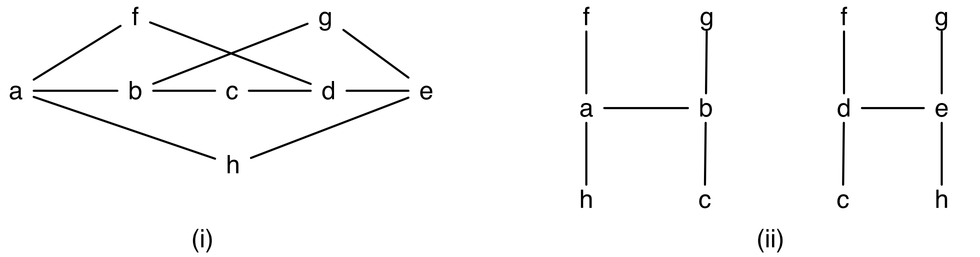

Figure 6 The model $\mathbb{H} \uplus \mathbb{H}$, and its edge-decomposition into two $\mathbb{H}$ graphs.

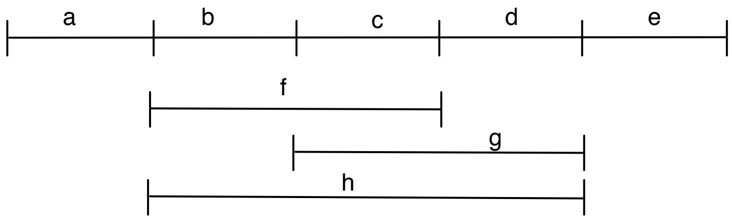

Figure 7 A depiction of the interval relations corresponding to an $\mathbb{H} \uplus \mathbb{H}$ graph.

Figure 6(ii) shows two examples of $\mathbb{H}$. Now $\mathbb{H}$ by itself is not the undirected graph for any structure in $\mathfrak{M}^{bounded\_meeting}$; however, we can use $\mathbb{H}$ to construct a graph that will play a critical role in the characterization of $\mathfrak{M}^{bounded\_meeting}$.

We first need to specify a way in which new graphs can be constructed from existing ones.

▶ **Definition 21.** A graph $\mathcal{G} = \langle V, E \rangle$ is edge-decomposable into a set of graphs $\mathcal{H}$ iff

1. $\mathcal{H}_i \subset \mathcal{G}$, for each $\mathcal{H}_i \in H$;
2. $E_i \cap E_j = \emptyset$, for each $\mathcal{H}_i = \langle V_i, E_i \rangle$ and $\mathcal{H}_j = \langle V_j, E_j \rangle$;
3. $E = \bigcup_i E_i$.

Thus, a graph $\mathcal{G}$ is edge-decomposable into a set of subgraphs iff the set of edges in $\mathcal{G}$ can be partitioned. Figure 6(i) depicts the graph $\mathbb{H} \uplus \mathbb{H}$; it is edge-decomposable into the two graphs in Figure 6(ii). We will use the notation $\mathcal{G} = \mathcal{H}_1 \uplus ... \uplus \mathcal{H}_n$ to indicate that $\mathcal{G}$ is edge-decomposable into $\mathcal{H}_1, ..., \mathcal{H}_n$.

Elsewhere in graph theory, $\mathbb{H} \uplus \mathbb{H}$ is the unique extremal graph of order 8, that is, it contains the maximal number of edges for a graph of girth[10] 5 on 8 vertices.

▶ **Lemma 22.** *If $\mathcal{M}$ is an orientation of $\mathbb{H} \uplus \mathbb{H}$, then $\mathcal{M} \in \mathfrak{M}^{bounded\_meeting}$.*

Figure 7 illustrates how $\mathbb{H} \uplus \mathbb{H}$ is related to the interval relations.

4.3 Characterization of $\mathfrak{M}^{bounded\_meeting}$

In the preceding section, we explicitly identified some graphs that are structures in $\mathfrak{M}^{bounded\_meeting}$, but we are ultimately interested in characterizing *all* such graphs.

[10] The girth of a graph G is the length of the shortest cycle in G.

▶ **Definition 23.** A graph G is triangle-free iff it does not contain any induced cycles of length 3 (i.e. G contains no induced K_3 subgraphs).

▶ **Definition 24.** A graph G is 2-connected iff there exist two vertex-disjoint paths between any two vertices in G.

A well-known result in graph theory [4] shows that a graph is 2-connected iff each pair of vertices are elements of the same cycle.

▶ **Definition 25.** The distance between two vertices in a graph G is the number of edges in a shortest path that connects the two vertices.

The diameter of a graph G (denoted by $diam(G)$) is the greatest distance between any two vertices of G.

▶ **Lemma 26.** *Suppose $\mathcal{M} \in \mathfrak{M}^{bounded\_meeting}$ and let G be the undirected graph for $\mathcal{M}$ such that G is not a short model.*

G is a 2-connected triangle-free graph such that $diam(G) = 3$.

Proof. G is triangle-free iff for any $\mathbf{x} \in V$, $N(\mathbf{x}) \cap N^2(\mathbf{x}) = \emptyset$, which is equivalent to condition (1) in Definition 10.

By condition (3) in Definition 10, there are at most two vertices between any $\mathbf{x}, \mathbf{y}$, so that $diam(G) = 3$ (and hence G is connected).

Suppose G is not 2-connected; since G is not a short model, there must exist $\mathbf{x}, \mathbf{y} \in V$ such that there is a unique P_4 subgraph with $\mathbf{x}$ and $\mathbf{y}$ as endpoints. However, this violates condition (2) in Definition 10. ◀

Note that the converse of this Lemma does not hold; we need to determine which 2-connected triangle-free graphs are structures in $\mathfrak{M}^{bounded\_meeting}$.

▶ **Theorem 27.** *Suppose $\mathcal{M}$ is the orientation of a twin-free graph.*

$\mathcal{M} \in \mathfrak{M}^{bounded\_meeting}$ iff $\mathcal{M}$ is the orientation of a short model or of a graph in which any two elements $\mathbf{x}, \mathbf{y}$ are elements of a subgraph $G \subseteq \mathcal{M}$ such that

$$G \cong \mathbb{H} \cup \mathbb{H}$$

Proof. (Sketch) By Lemma 26, any two elements $\mathbf{x}, \mathbf{y}$ are elements of an induced P_2, P_3, C_5, or C_6 subgraph.

By condition (2) in Definition 10, there exist additional elements $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3} \in V$ that create new C_5 subgraphs, so that the C_6 subgraph generates an $\mathbb{H} \cup \mathbb{H}$ subgraph that contains $\mathbf{x}, \mathbf{y}$. ◀

Finally, we can give a constructive characterization for finite structures in $\mathfrak{M}^{bounded\_meeting}$.

▶ **Definition 28.** Let G_1,G_2 be two graphs that each contain an induced P_k subgraph.

A graph obtained from G_1 and G_2 by identifying the two subgraphs is a P_k-gluing of G_1 and G_2.

▶ **Definition 29.** A full graph is a graph in which each P_4 subgraph is an edge cover of an induced C_5 subgraph.

▶ **Theorem 30.** *Suppose $\mathcal{M} \in \mathfrak{M}^{bounded\_meeting}$ such that $\mathcal{M}$ is twin-free and finite, and that G is the undirected graph for $\mathcal{M}$.*

G is not 2-connected iff $G \cong P_2$ or $G \cong P_3$.

G is 2-connected iff there exists a sequence $G_1, ..., G$ such that

1. *$G_1 \cong P_n$;*
2. *G is a full graph;*
3. *G_{i+1} is the result of P_4-gluing G_i and C_5.*

Proof. (Sketch)

By Lemma 19 and Lemma 26, G is connected but not cyclic iff $G \cong P_2$ or $G \cong P_3$.

Suppose G is 2-connected.

By Theorem 27, each pair of vertices are elements of an $\mathbb{H} \cup \mathbb{H}$ subgraph, so that G is a full graph.

Let P_k be the longest path in G. By condition (2) in Definition 10, every P_4 subgraph is an edge cover of an induced C_5 subgraph, which is equivalent to the P_4-gluing of a subgraph to C_5. ◀

Recall the original motivation for this work – since all applications of Allen's Interval Algebra consists in the specification of temporal constraints and the use of constraint satisfaction techniques to find solutions. Such solutions correspond to models of T^*_{allen}. Given the synonymy of T^*_{allen} and $T_{bounded\_meeting}$ (Theorem 7), these last two results not only give us a complete characterization of the finite models of $T_{bounded\_meeting}$ up to isomorphism, but they also give us a complete characterization of the solutions of a set of temporal constraints.

5 Summary

Constraint satisfaction with relational calculi such as Allen's Interval Algebra has been the predominant application of temporal concepts within commonsense reasoning. Yet in some way, this has diminished the role played by the different time ontologies that provide their foundations. It has long been known that the first-order theory of Allen's Interval Algebra is interpretable by certain ontologies of time intervals, in particular, the ontology $T_{interval\_meeting}$. This perspective has been considered sufficient for showing that Allen's Interval Algebra was in some sense sound with respect to its ontological foundations. In this paper, we have specified an ontology $T_{bounded\_meeting}$ that is weaker than $T_{interval\_meeting}$ and which is logically synonymous with Allen's Interval Algebra. Finally, we have provided a characterization of the models of $T_{bounded\_meeting}$ up to isomorphism, by first specifying a class of mathematical structures, and then showing that $T_{bounded\_meeting}$ axiomatizes this class of structures. This characterization gives us insights into the set of all possible solutions for a set of temporal constraints that can be specified by Allen's Interval Algebra.

The next step in this direction is a full characterization of the infinite models of $T_{bounded\_meeting}$, which would provide an alternative characterization of the models of $T_{interval\_meeting}$. It would also enable us to explore extensions of $T_{bounded\_meeting}$ that axiomatize dense orderings. This is equivalent to revisiting Hayes' and Ladkins description of models with respect to intervals on $\mathbb{Q}$ and $\mathbb{Z}$. Formalize these results as representation theorems.

Another major question is the relationship of $T_{bounded\_meeting}$ to other ontologies of time intervals, such as periods [12], in which the primitive relations are precedence and inclusion rather than meets. In particular, this would require a characterization of the mereology of intervals that is definable within models of $T_{interval\_meeting}$.

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