# Verification of Time Ontologies with Points and Intervals

Michael Grüninger
Darren Ong
Department of Mechanical and Industrial Engineering
University of Toronto
Toronto, Ontario, Canada M5S 3G8

## Abstract

*Ontology verification is concerned with the relationship between the intended structures for an ontology and the models of the axiomatization of the ontology. The verification of a particular ontology requires characterization of the models of the ontology up to isomorphism and a proof that these models are equivalent to the intended structures for the ontology. In this paper we consider the verification of three time ontologies (first introduced by Hayes in his Catalog of Temporal Theories) that axiomatize both timepoints and time intervals together with the relationships between them. We identify axioms that are missing from these ontologies and provide a complete account of the metatheoretic relationships between the ontologies.*

## I. Introduction

An ontology is a logical theory that axiomatizes the concepts in some domain, which can either be commonsense knowledge representation (such as time, process, and shape) or the representation of knowledge in more technical domains (such as biology and engineering). Verification of an ontology is concerned with the relationship between the intended structures for an ontology and the models of the axiomatization of the ontology. In particular, we want to characterize the models of an ontology up to isomorphism and determine whether or not these models are equivalent to the intended structures for the ontology. This relationship between the intended structures and the models of the axiomatization plays a key role in the application of ontologies in areas such as semantic integration and decision support.

Over the years, a number of first-order ontologies for time have been proposed, and many of these were collected by Hayes in [1]. In addition to ontologies that axiomatize only time points and ontologies that axiomatize only time intervals ([2], [3]), Hayes also included three ontologies that axiomatize both timepoints and time intervals together with the relationships between them. Remarkably, there has been no work on a formal characterization of the models of these ontologies up to isomorphism, despite the fact that upper ontologies such as SUMO [4], Cyc [5], and OWL-Time ([6], [7], [8]) incorporate axiomatizations that are very similar to the ontologies from [1]. Although Hayes describes some models of the ontologies, he does not provide a characterization of *all* models, or any discussion concerning the existence of potentially unintended models.

After reviewing our approach to ontology verification, we examine the three ontologies from [1] that combine timepoints and time intervals – $endpoints$, $vector\_continuum$, and $point\_continuum$ [1]. For each theory, we provide a representation theorem (characterization of the models of the ontology up to isomorphism) which also enables us to understand the metalogical relationships between the theories. Within each theory, we also identify missing axioms that are needed to either derive claims made by Hayes or to eliminate unintended models and ensure the proof of the representation theorem for the theory.

All proofs in this paper were generated using the Prover9 automated theorem prover. Counterexamples (models that satisfy the negation of a proposed axiom and hence demonstrate the independence of an axiom from an ontology) were generated using Mace4.

In this paper, we are providing a logical analysis of the models of each ontology; we do not critique the underlying ontological commitments, and hence we are not interested in questions concerning which ontology is appropriate to

[1]The online CLIF (Common Logic Interchange Format) axiomatization of these theories can be found at
`http://stl.mie.utoronto.ca/colore/time/endpoints`
`http://stl.mie.utoronto.ca/colore/time/vector_continuum`
`http://stl.mie.utoronto.ca/colore/time/point_continuum`

use as a time ontology in a given context.

# II. Ontology Verification

Our methodology revolves around the application of model-theoretic notions to the design and analysis of ontologies. The semantics of the ontology's terminology can be characterized by a set of structures, which we refer to as the set of intended structures for the ontology. Intended structures are specified with respect to the models of well-understood mathematical theories (such as partial orderings, lattices, incidence structures, geometries, and algebra). The extensions of the relations in an intended structure are then specified with respect to properties of these models.

Why do we care about ontology verification? The relationship between the intended models and the models of the axiomatization plays a key role in the application of ontologies in areas such as semantic integration and decision support. Software systems are semantically integrated if their sets of intended models are equivalent. In the area of decision support, the verification of an ontology allows us to make the claim that any inferences drawn by a reasoning engine using the ontology are actually entailed by the ontology's intended models. If an ontology's axiomatization has unintended models, then it is possible to find sentences that are entailed by the intended models, but which are not provable from the axioms of the ontology. The existence of unintended models also prevents the entailment of sentences or a possible barriers to interoperability.

With ontology verification, we want to characterize the models of an ontology up to isomorphism and determine whether or not these models are elementarily equivalent to the intended structures of the ontology. From a mathematical perspective this is formalized by the notion of representation theorems. Representation theorems are proven in two parts – we first prove every intended structure is a model of the ontology and then prove that every model of the ontology is elementary equivalent to some intended structure. Classes of structures for theories within an ontology are therefore axiomatized up to elementary equivalence – the theories are satisfied by any structure in the class, and any model of the theories is elementarily equivalent to a structure in the class.

## A. Interpretability

We now show how a theorem about the relationship between the class of the ontology's models and the class of intended structures can be replaced by a theorem about the relationship between the ontology (a theory) and the theory axiomatizing the intended structures (assuming that such axiomatization is known). We will later show how we can use automated reasoners to prove this relationship and thus verify an ontology in a (semi-)automated way.

The relationship between theories $T_A$ and $T_B$ is the notion of interpretation, which is a mapping from the language of $T_A$ to the language of $T_B$ that preserves the theorems of $T_A$. We adopt the next two definitions from [9]:

*Definition 1:* An interpretation $\pi$ of a theory $T_0$ with language $L_0$ into a theory $T_1$ with language $L_1$ is a function on the set of parameters of $L_0$ such that

1) $\pi$ assigns to $\forall$ a formula $\pi_\forall$ of $L_1$ in which at most the variable $v_1$ occurs free, such that

$$T_1 \models (\exists v_1)\, \pi_\forall$$

2) $\pi$ assigns to each n-place relation symbol $P$ a formula $\pi_P$ of $L_1$ in which at most the variables $v_1, ..., v_n$ occur free.
3) For any sentence $\sigma$ in $L_0$,

$$T_0 \models \sigma \Rightarrow T_1 \models \pi(\sigma)$$

Thus, the mapping $\pi$ is an interpretation of $T_0$ if it preserves the theorems of $T_0$.

If there is an interpretation of $T_A$ in $T_B$, then there exists a set of sentences (referred to as translation definitions) in the language $L_A \cup L_B$ of the form

$$(\forall \overline{x})\, p_i(\overline{x}) \equiv \varphi(\overline{x})$$

where $p_i(\overline{x})$ is a relation symbol in $L_A$ and $\varphi(\overline{x})$ is a formula in $L_B$.

We will say that two theories $T_A$ and $T_B$ are definably equivalent iff they are mutually interpretable, i.e. $T_A$ is interpretable in $T_B$ and $T_B$ is interpretable in $T_A$.

*Definition 2:* An interpretation $\pi$ of a theory $T_0$ into a theory $T_1$ is faithful iff there exists an interpretation $\pi$ of $T_0$ into $T_1$ and

$$T_0 \not\models \sigma \Rightarrow T_1 \not\models \pi(\sigma)$$

for any sentence $\sigma \in \mathcal{L}(T_0)$.

Thus, the mapping $\pi$ is a faithful interpretation of $T_0$ if it preserves satisfiability with respect to $T_0$. We will also refer to this by saying that $T_0$ is faithfully interpretable in $T_1$.

## B. Representation Theorems

The primary challenge for someone attempting to prove representation theorems is to characterize the models of an ontology up to isomorphism.

*Definition 3:* A class of structures $\mathfrak{M}$ can be represented by a class of structures $\mathfrak{N}$ iff there is a bijection $\varphi : \mathfrak{M} \rightarrow \mathfrak{N}$ such that for any $\mathcal{M} \in \mathfrak{M}$, $\mathcal{M}$ is definable in $\varphi(\mathcal{M})$ and $\varphi(\mathcal{M})$ is definable in $\mathcal{M}$.

The key to using theorem proving and model finding to support ontology verification is the following theorem ([10]):

*Theorem 1:* A theory $T_1$ is definably equivalent with a theory $T_2$ iff the class of models $Mod(T_1)$ can be represented by $Mod(T_2)$.

Let $\mathfrak{M}^{intended}$ be the class of intended structures for the ontology, and let $T_{onto}$ be the axiomatization of the ontology. The necessary direction of a representation theorem (i.e. if a structure is intended, then it is a model of the ontology's axiomatization) can be stated as

$$\mathcal{M} \in \mathfrak{M}^{intended} \Rightarrow \mathcal{M} \in Mod(T_{onto})$$

If we suppose that the theory that axiomatizes $\mathfrak{M}^{intended}$ is the union of some previously known theories $T_1, ..., T_n$, then by Theorem 1 we need to show that $T_{onto}$ interprets $T_1 \cup ... \cup T_n$. If $\Delta$ is the set of translation definitions for this interpretation, then the necessary direction of the representation theorem is equivalent to the following reasoning task:

$$T_{onto} \cup \Delta \models T_1 \cup ... \cup T_n \qquad \textbf{(Rep-1)}$$

The sufficient direction of a representation theorem (any model of the ontology's axiomatization is also an intended structure) can be stated as

$$\mathcal{M} \in Mod(T_{onto}) \Rightarrow \mathcal{M} \in \mathfrak{M}^{intended}$$

In this case, we need to show that $T_1 \cup ... \cup T_n$ interprets $T_{onto}$. If $\Pi$ is the set of translation definitions for this interpretation, the sufficient direction of the representation theorem is equivalent to the following reasoning task:

$$T_1 \cup ... \cup T_n \cup \Pi \models T_{onto} \qquad \textbf{(Rep-2)}$$

## III. Graphical Incidence Structures

Before we begin the model-theoretic analysis of the time ontologies, we introduce the classes of mathematical structures which will be used in the representation theorems.

The basic building blocks for the models presented in this paper are based on the notion of incidence structures ([11]).

*Definition 4:* A k-partite incidence structure is a tuple $\mathbb{I} = (\Omega_1, ..., \Omega_k, \mathbf{in})$, where $\Omega_1, ..., \Omega_k$ are sets with

$$\Omega_i \cap \Omega_j = \emptyset, i \neq j$$

and

$$\mathbf{in} \subseteq (\bigcup_{i \neq j} \Omega_i \times \Omega_j)$$

Two elements of $\mathbb{I}$ that are related by **in** are called incident.

The models of the time ontologies in this paper will be constructed using special classes of incidence structures.

*Definition 5:* An strict graphical incidence structure is a bipartite incidence structure

$$\mathbb{G} = \langle X, Y, \mathbf{in}^{\mathbf{G}} \rangle$$

such that all elements of $Y$ are incident with exactly two elements of $X$, and for each pair of points $\mathbf{p}, \mathbf{q} \in X$ there exists a unique element in $Y$ that is incident with both $\mathbf{p}$ and $\mathbf{q}$.

The class of strict graphical incidence structures is axiomatized by $T_{strict\_graphical}$ [2].

*Definition 6:* An strong graphical incidence structure is a bipartite incidence structure

$$\mathbb{S} = \langle X, Y, \mathbf{in}^{\mathbf{S}} \rangle$$

such that all elements of $Y$ are incident with either one or two elements of $X$, and for each pair of points $\mathbf{p}, \mathbf{q} \in X$ there exists a unique element in $Y$ that is incident with both $\mathbf{p}$ and $\mathbf{q}$.

The class of strong graphical incidence structures is axiomatized by $T_{strong\_graphical}$ [3].

These two classes of incidence structures get their names from graph-theoretic representation theorems of their own.

*Definition 7:* A graph $G = (V, E)$ consists of a nonempty set $V$ of vertices and a set $E$ of ordered pairs of vertices called edges.

An edge whose vertices coincide is called a loop. A graph with no loops or multiple edges is a simple graph.

A complete graph is a graph in which each pair of vertices is adjacent.

*Theorem 2:* Let $G = (V, E)$ be a complete simple graph.

A bipartite incidence structure is a strict graphical incidence structure iff it is isomorphic to $\mathbb{I} = (V, E, \in)$, where $\in$ is the containment relation for vertices in an edge.

*Theorem 3:* Let $G = (V, E)$ be a complete graph with loops.

A bipartite incidence structure is a strong graphical incidence structure iff it is isomorphic to $\mathbb{I} = (V, E, \in)$.

These two representation theorems show that there is a one-to-one correspondence between each class of incidence structures and the given class of graphs, and in so doing, we have a characterization of the incidence structures up to isomorphism.

The third class of incidence structures used in this paper require the notion of the direct product of incidence structures:

*Definition 8:* Given two incidence structures

[2] http://stl.mie.utoronto.ca/colore/incidence/strict-graphical.clif

[3] http://stl.mie.utoronto.ca/colore/incidence/strong-graphical.clif

$\mathcal{I}_1 = \langle \mathcal{P}_1, \mathcal{L}_1, \mathbf{in}_1 \rangle$ and $\mathcal{I}_2 = \langle \mathcal{P}_2, \mathcal{L}_2, \mathbf{in}_2 \rangle$ the direct product $\mathcal{I}_1 \times \mathcal{I}_2$ is the incidence structure such that

- $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$;
- $\mathcal{L} = (\mathcal{P}_1 \times \mathcal{L}_2) \cup (\mathcal{L}_1 \times \mathcal{P}_2)$;
- the point $(x, y)$ is incident with the line $(x, L) \in \mathcal{P}_1 \times \mathcal{L}_2$ iff $\langle y, L \rangle \in \mathbf{in}_2$;
- the point $(x, y)$ is incident with the line $(M, y) \in \mathcal{L}_1 \times \mathcal{P}_2$ iff $\langle x, M \rangle \in \mathbf{in}_1$.

*Definition 9:* Let $\mathbb{G} = \langle P, G, \mathbf{in}^{\mathbf{G}} \rangle$ be a strict graphical incidence structure. Let $\mathbb{S} = \langle P, S, \mathbf{in}^{\mathbf{S}} \rangle$ be a strong graphical incidence structure.

An incidence structure $\mathbb{D} = \langle P, X, \mathbf{in}^{\mathbf{D}} \rangle$ is a double complete incidence structure iff

$$\mathbb{D} \cong \mathbb{G} \times \mathbb{S}$$

The class of double complete incidence structures is axiomatized by $T_{double\_complete}$ [4].

As above, we can also provide a representation theorem for double complete incidence structures with respect to a class of graphs:

*Theorem 4:* Let $G = (V, E)$ be a complete graph in which $E$ is a symmetric reflexive relation.

A bipartite incidence structure is a double complete incidence structure iff it is isomorphic to $\mathbb{I} = (V, E, \in)$.

## IV. Endpoints

The *endpoints* theory combines the language of intervals and points by defining the functions *beginof*, *endof*, and *between* to relate intervals to points and vice-versa. This theory imports the axioms of *linear_point* that define the binary *before* relation between timepoints as transitive and irreflexive, and impose the condition that all timepoints are linearly ordered and infinite in both directions. The *endpoints* theory includes axioms defining the *meets-at* relation as one between two intervals and the point at which they meet along with conservative definitions for *meets*, *precedes*, *overlaps*, *starts*, *during* and *finishes*. Finally, an axiom that restricts the *beginof* an interval to always come *before* its *endof* and another that states that intervals are *between* two points if they are properly ordered complete the theory. The first of the final two axioms has the effect of preventing single-point intervals from existing in this theory as an interval that has the same point as its *beginof* and *endof* would be inconsistent due to the irreflexive property of the *before* relation.

In order to prove the representation theorems below, we discovered that the following axioms are missing from *endpoints* as presented in [1]:

$$(\forall x)\, timepoint(x) \vee timeinterval(x) \tag{1}$$

$$(\forall x)\, timepoint(x) \supset \neg timeinterval(x) \tag{2}$$

$$(\forall x, y)\, before(x, y) \supset timeinterval(between(x, y)) \tag{3}$$

Let $T_{endpoints}$ be the theory in which *endpoints* is extended with these axioms.

The first step in the verification of $T_{endpoints}$ is to prove the reasoning tasks that instantiate $(\mathbf{Rep-1})$ and $(\mathbf{Rep-2})$:

*Theorem 5:* $T_{endpoints}$ is definably equivalent to

$$T_{linear\_ordering} \cup T_{strict\_graphical}$$

*Proof:* Let $\Delta$ be the following set of translation definitions:

$$(\forall x)\, point(x) \equiv timepoint(x)$$

$$(\forall x)\, line(x) \equiv timeinterval(x)$$

$$(\forall x, y)\, in(x, y) \equiv ((beginof(y) = x) \vee (endof(y) = x))$$

$$(\forall x, y)\, before(x, y) \equiv leq(x, y)$$

Using Prover9, we have shown that

$$T_{endpoints} \cup \Delta \models T_{linear\_ordering} \cup T_{strict\_graphical}$$

Let $\Pi$ be the following set of translation definitions:

$$(\forall x)\, timepoint(x) \equiv point(x)$$

$$(\forall x)\, timeinterval(x) \equiv line(x)$$

$$(\forall x, y)(beginof(y) = x) \equiv ((in(x, y) \wedge ((\forall z) in(z, y) \supset leq(x, z))$$

$$(\forall x, y)(endof(y) = x) \equiv ((in(x, y) \wedge ((\forall z) in(z, y) \supset leq(z, x))$$

$$(\forall x, y)\, before(x, y) \equiv leq(x, y)$$

Using Prover9, we have shown that

$$T_{linear\_ordering} \cup T_{strict\_graphical} \cup \Pi \models T_{endpoints}$$

■

The second step in the verification of $T_{endpoints}$ is to define the class of intended models:

*Definition 10:* $\mathfrak{M}^{endpoints}$ is the following class of structures: $\mathcal{M} \in \mathfrak{M}^{endpoints}$ iff

1) $\mathcal{M} \cong \mathcal{P} \cup \mathbb{G}$, where
   a) $\mathcal{P} = \langle P, \leq \rangle$ is a linear ordering
   b) $\mathbb{G} = \langle P, G, \mathbf{in}^{\mathbf{G}} \rangle$ is a strict graphical incidence structure.
2) $\langle \mathbf{t} \rangle \in \mathbf{timepoint}$ iff $\mathbf{t} \in P$;
3) $\langle \mathbf{i} \rangle \in \mathbf{timeinterval}$ iff $\mathbf{i} \in G$;

[4] http://stl.mie.utoronto.ca/colore/incidence/double-complete.clif

4) $\mathbf{beginof}(\mathbf{i}) = \mathbf{t}$ iff $\langle \mathbf{t}, \mathbf{i} \rangle \in \mathbf{in^G}$ and for any $\mathbf{t}' \in P$ such that $\langle \mathbf{t}', \mathbf{i} \rangle \in \mathbf{in^G}$, we have $\mathbf{t} \leq \mathbf{t}'$.
5) $\mathbf{endof}(\mathbf{i}) = \mathbf{t}$ iff $\langle \mathbf{t}, \mathbf{i} \rangle \in \mathbf{in^G}$ and for any $\mathbf{t}' \in P$ such that $\langle \mathbf{t}', \mathbf{i} \rangle \in \mathbf{in^G}$, we have $\mathbf{t}' \leq \mathbf{t}$.
6) $\mathbf{between}(\mathbf{t_1}, \mathbf{t_2}) = \mathbf{i}$ iff $\langle \mathbf{t_1}, \mathbf{i} \rangle$, $\langle \mathbf{t_2}, \mathbf{i} \rangle \in \mathbf{in^G}$;
7) $\langle \mathbf{t_1}, \mathbf{t_2} \rangle \in \mathbf{before}$ iff $\mathbf{t_1} < \mathbf{t_2}$.

We can now state the Representation Theorem for $T_{endpoints}$:

*Theorem 6:* $\mathcal{M} \in \mathfrak{M}^{endpoints}$ iff $\mathcal{M} \in Mod(T_{endpoints})$.

*Proof:* This follows from Theorem 5 and Theorem 1, together with the fact that $T_{strict\_graphical}$ axiomatizes the class of strict graphical incidence structures and $T_{linear\_ordering}$ axiomatizes the class of linear orderings. ■

## V. Vector Continuum

The *vector_continuum* theory is a theory of timepoints and intervals that introduces the notion of orientation of intervals. It also imports theory *linear_point* and adds to it axioms that define the *meets-at* relation and the conservative definitions of *meets*, *precedes*, *overlaps*, *starts*, *during* and *finishes* in the same way as the *endpoints* theory. Although it has the same three functions (*beginof*, *endof*, and *between*) that transform intervals to points and vice-versa, it differs in its definition of *between* by allowing the formation of intervals whose *endof* point is equal to or before its *beginof*. Thus, unlike the *endpoints* theory, every interval in *vector_continuum* has a reflection in the opposite direction via the *back* function and intervals oriented in the forward direction (*beginof* is *before endof*) are defined by the *forward* relation. In this theory single-point intervals, known as *moments*, are defined as intervals whose *beginof* and *endof* points are the same.

Similar to *endpoints*, we discovered that the following axioms are missing from *vector_continuum* as presented in [1]:

$$(\forall x)\, timepoint(x) \vee timeinterval(x) \quad (4)$$

$$(\forall i, p)\, (beginof(i) = p) \wedge endof(i) = q \supset$$
$$between(p, q) = i \quad (5)$$

Without these axioms, there exist models that falsify the sentence

$$(\forall i)\, (back(back(i)) = i)$$

Hence, this sentence is not provable from *vector_continuum*, as Hayes claims.

Let $T_{vc}$ be the theory in which *vector_continuum* is extended with these axioms.

*Theorem 7:* $T_{vc}$ is definably equivalent to $T_{linear\_ordering} \cup T_{double\_complete}$

*Proof:* Let $\Delta$ be the following set of translation definitions:

$$(\forall x)\, point(x) \equiv timepoint(x)$$
$$(\forall x)\, line(x) \equiv timeinterval(x)$$
$$(\forall x, y)\, in^g(x, y) \equiv ((beginof(y) = x) \vee (endof(y) = x))$$
$$(\forall x, y)\, in^s(x, y) \equiv ((beginof(y) = x) \vee (endof(y) = x))$$
$$(\forall x, y)\, before(x, y) \equiv leq(x, y)$$

Using Prover9, we have shown that

$$T_{vc} \cup \Delta \models T_{linear\_ordering} \cup T_{double\_complete}$$

Let $\Pi$ be the following set of translation definitions:

$$(\forall i, t)\, (beginof(i) = t) \equiv$$
$$(in^g(t, i) \wedge ((\forall t')\, in^g(t', i) \supset leq(t, t'))$$
$$\vee (in^s(t, i) \wedge ((\forall t')\, in^s(t', i) \supset leq(t', t))$$
$$(\forall i, t)\, (endof(i) = t) \equiv$$
$$(in^g(t, i) \wedge ((\forall t')\, in^g(t', i) \supset leq(t', t))$$
$$\vee (in^s(t, i) \wedge ((\forall t')\, in^s(t', i) \supset leq(t, t'))$$

Using Prover9, we have shown that

$$T_{linear\_ordering} \cup T_{double\_complete} \cup \Pi \models T_{vc}$$

■

The definition of the class of intended structures is slightly more complicated since we need to use two different incidence substructures – a strict graphical incidence structure for forward intervals and a strong graphical incidence structure for backward intervals:

*Definition 11:* $\mathfrak{M}^{vc}$ is the following class of structures: $\mathcal{M} \in \mathfrak{M}^{vc}$ iff

1) $\mathcal{M} \cong \mathcal{P} \cup (\mathbb{G} \times \mathbb{S})$, where
   a) $\mathcal{P} = \langle P, \leq \rangle$ is a linear ordering;
   b) $\mathbb{G} = \langle P, G, \mathbf{in^G} \rangle$ is a strict graphical incidence structure;
   c) $\mathbb{S} = \langle P, S, \mathbf{in^S} \rangle$ is a strong graphical incidence structure.
2) $\langle \mathbf{t} \rangle \in \mathbf{timepoint}$ iff $\mathbf{t} \in P$;
3) $\langle \mathbf{i} \rangle \in \mathbf{timeinterval}$ iff $\mathbf{i} \in G \cup S$;
4) $\langle \mathbf{i} \rangle \in \mathbf{moment}$ iff $\mathbf{i} \in S$ and there exists a unique $\mathbf{t} \in P$ such that $\langle \mathbf{t}, \mathbf{i} \rangle \in \mathbf{in^S}$ ;
5) $\mathbf{beginof}(\mathbf{i}) = \mathbf{t}$ iff
   - $\langle \mathbf{t}, \mathbf{i} \rangle \in \mathbf{in^G}$ and for any $\mathbf{t}' \in P$ such that $\langle \mathbf{t}', \mathbf{i} \rangle \in \mathbf{in^G}$, we have $\mathbf{t} \leq \mathbf{t}'$, or
   - $\langle \mathbf{t}, \mathbf{i} \rangle \in \mathbf{in^S}$ and for any $\mathbf{t}' \in P$ such that $\langle \mathbf{t}', \mathbf{i} \rangle \in \mathbf{in^S}$, we have $\mathbf{t}' \leq \mathbf{t}$.

6) $\mathbf{endof(i) = t}$ iff
- $\langle \mathbf{t}, \mathbf{i} \rangle \in \mathbf{in^G}$ and for any $\mathbf{t'} \in P$ such that $\langle \mathbf{t'}, \mathbf{i} \rangle \in \mathbf{in^G}$, we have $\mathbf{t'} \leq \mathbf{t}$, or
- $\langle \mathbf{t}, \mathbf{i} \rangle \in \mathbf{in^S}$ and for any $\mathbf{t'} \in P$ such that $\langle \mathbf{t'}, \mathbf{i} \rangle \in \mathbf{in^S}$, we have $\mathbf{t} \leq \mathbf{t'}$.

7) $\mathbf{between(t_1, t_2) = i}$ iff $\langle \mathbf{t_1}, \mathbf{i} \rangle, \langle \mathbf{t_1}, \mathbf{i} \rangle \in \mathbf{in^G} \cup \mathbf{in^S}$.

The Representation Theorem for $T_{vc}$ shows that this definition of intended structures does characterize the models of $T_{vc}$ up to isomorphism:

*Theorem 8:* $\mathcal{M} \in \mathfrak{M}^{vc}$ iff $\mathcal{M} \in Mod(T_{vc})$.

*Proof:* This follows from Theorem 7 and Theorem 1, together with the fact that $T_{double\_complete}$ axiomatizes the class of double complete incidence structures and $T_{linear\_ordering}$ axiomatizes the class of linear orderings. ■

## VI. Relationship between $T_{endpoints}$ and $T_{vc}$

The $T_{endpoints}$ and $T_{vc}$ theories have the same primitive nonlogical lexicon, and hence we can determine their relationship using the notions of satisfiability, extension, and independence. In particular, we use the following notion:

*Definition 12:* Let $T_1$ and $T_2$ be theories with the language $\mathcal{L}$. The similarity of $T_1$ and $T_2$ is the strongest subtheory of $T_1$ and $T_2$ so that for all $\sigma, \phi \in \mathcal{L}$:
if $T_1 \models \sigma$ and $T_2 \models \phi$ and $T \not\models \sigma$ and $T \not\models \phi$, then either $\sigma \vee \phi$ is independent of $T$ or it is a tautology.

Let $Sim(endpoint, vc)$ be the theory which is equivalent to $T_{vc}$ with the axiom

$$(\forall p, q)\, timepoint(p) \wedge timepoint(q) \supset$$
$$(beginof(between(p, q)) = p)$$
$$\wedge endof(between(p, q)) = q)$$

replaced by

$$(\forall p, q)\, before(p, q) \supset$$
$$(beginof(between(p, q)) = p)$$
$$\wedge endof(between(p, q)) = q)$$

*Theorem 9:* $Sim(endpoint, vc)$ is the similarity of $T_{endpoints}$ and $T_{vc}$.

*Proof:* Let $backwards$ be the sentence

$$(\forall i_1)\, timeinterval(i_1) \supset (\exists i_2)\, timeinterval(i_2)$$
$$\wedge (beginof(i_2) = endof(i_1)) \wedge (endof(i_2) = beginof(i_1))$$

Let $no\_backwards$ be the sentence

$$(\forall i_2)\, timeinterval(i_1) \supset \neg(\exists i_2)\, timeinterval(i_2)$$
$$\wedge (beginof(i_2) = endof(i_1)) \wedge (endof(i_2) = beginof(i_1))$$

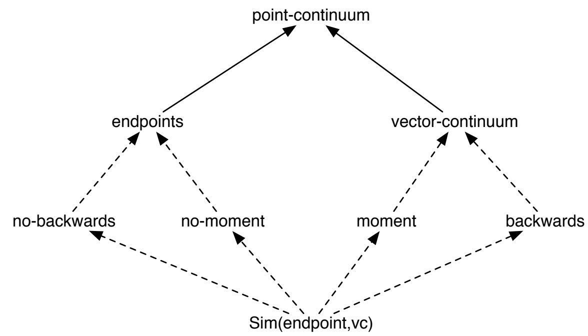


**Figure 1. Relationships between the time ontologies for points and intervals. Dotted lines denote nonconservative extension and solid lines denote faithful interpretation.**

Let $moment$ be the sentence

$$(\forall t)\, timepoint(t) \supset (\exists i)\, timeinterval(i)$$
$$\wedge (beginof(i) = t) \wedge (endof(i) = t)$$

Let $no\_moment$ be the sentence

$$(\forall t)\, timepoint(t) \supset \neg(\exists i)\, timeinterval(i)$$
$$\wedge (beginof(i) = t) \wedge (endof(i) = t)$$

Using Prover9, we have shown

$$T_{endpoints} \models no\_backwards \wedge no\_moment$$
$$T_{vector\_continuum} \models backwards \wedge moment$$

Using Mace4, we have shown that if disjunctions of these sentences are not tautologies, then they are independent of $Sim(endpoints, vc)$. ■

*Theorem 10:* $Sim(endpoint, vc)$ is definably equivalent to the theory $\mathcal{IQ}$ in [12].

The next corollary is not explicitly stated in [1], but it follows from the propositions used in the proof of Theorem 9.

*Corollary 1:* $T_{endpoints}$ and $T_{vc}$ are mutually inconsistent.

The relationships between the theories is summarized in Figure 1.

## VII. Point Continuum

The point-continuum theory combines intervals and points by defining the relation *in* that relates a point to the interval it is contained in. All intervals of this theory are oriented in the forward direction and are considered either *open*, when the *beginof* and *endof* points are not *in* in the interval, or *closed*, when the *beginof* and *endof* points are included *in* in the interval. Therefore, the axioms defining the functions *beginof*, *endof*, and *between* also make the distinction between open and closed intervals. The axiom for the function *closure* ensures that every *open* interval has a *closed* interval with the same endpoints. The relation *acoao* (also closed or also open) that compares two intervals is essential for the conservative definitions of the temporal relations *meets*, *starts* and *finishes*. With the distinction between *closed* and *open* intervals, *open* intervals in this theory can only meet and interval that is *closed*. Therefore, if two *open* intervals share an endpoint in common (where the *endof* one is equal to the *beginof* the other) these intervals do not *meet* each other, but instead they each meet the same *closed* single-point interval known as a *moment* that resides between them.

The following axioms are missing from $point\_continuum$:

$$(\forall x)\, timepoint(x) \vee timeinterval(x) \tag{6}$$

$$(\forall i)\, open(i) \supset timeinterval(i) \tag{7}$$

$$(\forall i)\, closed(i) \supset timeinterval(i) \tag{8}$$

$$(\forall i)\, timeinterval(i) \supset$$

$$timepoint(beginof(i)) \wedge timepoint(endof(i)) \tag{9}$$

Let $T_{pc}$ be the theory in which $point\_continuum$ is extended with these axioms.

### A. Relationship to the other Theories

$T_{pc}$ uses a language that expands the nonlogical lexicon of both $T_{endpoints}$ and $T_{vc}$, so we need to turn to the metatheoretic notion of relative interpretation to understand the relationships among these theories [5].

*Theorem 11:* $T_{pc}$ faithfully interprets $T_{endpoints}$.

*Proof:* If $\Delta$ is the following set of translation definitions:

$$(\forall x)\, timepoint^{ec}(x) \equiv timepoint^{pc}(x)$$

$$(\forall x)\, timeinterval^{ec}(x) \equiv open^{pc}(x)$$

$$(\forall x, y)\, (beginof^{ec}(x) = y) \equiv (beginof^{pc}(x) = y)$$

$$(\forall x, y)\, (endof^{ec}(x) = y) \equiv (endof^{pc}(x) = y)$$

then $T_{pc} \cup \Delta \models T_{endpoints}$. ■

*Theorem 12:* $T_{point\_continuum}$ faithfully interprets $T_{vc}$.

*Proof:* If $\Delta$ is the following set of translation definitions:

$$(\forall x)\, timepoint^{vc}(x) \equiv timepoint^{pc}(x)$$

[5]Since these theories use relations with the same names, we distinguish them by a superscript that denotes the theory in which they are axiomatized.

$$(\forall x)\, timeinterval^{vc}(x) \equiv closed^{pc}(x)$$

$$(\forall x, y)\, (beginof^{vc}(x) = y) \equiv (beginof^{pc}(x) = y)$$

$$(\forall x, y)\, (endof^{vc}(x) = y) \equiv (endof^{pc}(x) = y)$$

then $T_{pc} \cup \Delta \models T_{vc}$.

■

Notice that intervals in $T_{endpoints}$ are interpreted as open intervals in $T_{pc}$ and intervals in $T_{vc}$ are interpreted as closed intervals. In this sense, the the inconsistency between $T_{endpoints}$ and $T_{vc}$ appears as the disjointness of the classes of open and closed intervals in $T_{pc}$.

### B. Representation Theorem for $T_{pc}$

As the relationships between the three theories indicate, the verification of $T_{pc}$ combines the representation theorems for both $T_{endpoints}$ and $T_{vc}$.

*Theorem 13:* $T_{pc}$ is definably equivalent to $T_{linear\_ordering} \cup T_{strict\_graphical} \cup T_{double\_complete}$

*Proof:* We can use the translation definitions in Theorems 11 and 12 and the representation theorems for $T_{endpoints}$ (Theorem 6) and for $T_{vc}$ (Theorem 8) to show that

$$T_{pc} \cup \Delta \models$$

$$T_{linear\_ordering} \cup T_{strict\_graphical} \cup T_{double\_complete}$$

For the other direction, we can use the translation definitions from Theorems 5 and 7. ■

The characterization of the models of $T_{pc}$ combines the classes of models that were introduced in Definitions 10 and 11:

*Definition 13:* $\mathfrak{M}^{pc}$ is the following class of structures: $\mathcal{M} \in \mathfrak{M}^{pc}$ iff

1) $\mathcal{M} \cong \mathcal{P} \cup \mathbb{O} \cup (\mathbb{G} \times \mathbb{S})$, where
    a) $\mathcal{P} = \langle P, \leq \rangle$ is a linear ordering,
    b) $\mathbb{O} = \langle P, O, \mathbf{in^O} \rangle$ is a strict graphical incidence structure,
    c) $\mathbb{G} = \langle P, G, \mathbf{in^G} \rangle$ is a strict graphical incidence structure,
    d) $\mathbb{S} = \langle P, S, \mathbf{in^S} \rangle$ is a strong graphical incidence structure;
2) $\langle \mathbf{t} \rangle \in \mathbf{timepoint}$ iff $\mathbf{t} \in P$;
3) $\langle \mathbf{i} \rangle \in \mathbf{timeinterval}$ iff $\mathbf{i} \in O \cup G \cup S$;
4) $\langle i \rangle \in \mathbf{open}$ iff $\mathbf{i} \in O$;
5) $\langle i \rangle \in \mathbf{closed}$ iff $\mathbf{i} \in G \cup S$;

*Theorem 14:* $\mathcal{M} \in \mathfrak{M}^{pc}$ iff $\mathcal{M} \in Mod(T_{pc})$.

*Proof:* This follows from Theorem 13 and Theorem 1, together with the fact that $T_{strict\_graphical}$ axiomatizes the class of strict graphical incidence structures, $T_{double\_complete}$ axiomatizes the class of double complete incidence structures and $T_{linear\_ordering}$ axiomatizes the class of linear orderings. ■

## VIII. Summary

Three time ontologies first introduced by Hayes in [1] ($endpoints$, $vector\_continuum$, and $point\_continuum$) axiomatize both timepoints and timeintervals. We have provided a characterization of the intended structures for these ontologies up to isomorphism, and we have shown that these intended structures are isomorphic to the models of the ontologies. This characterization constitutes the verification of the ontologies.

In the course of proving the representation theorems, we identified axioms that were missing from each theory, and hence allowed unintended models. We also provided a complete characterization of the relationships among the theories. In particular, we specified the the similarity between $endpoints$ and $vector\_continuum$, and demonstrated that $point\_continuum$ can faithfully interpret the other two theories.

We can extend this methodology of ontology verification to other upper ontologies (such as SUMO, Cyc, and OWL-Time) that use similar first-order temporal theories for timepoints and intervals.

## References


[1] P. Hayes, "Catalog of temporal theories," University of Illinois Urbana-Champagne, Tech. Rep. Technical Report UIUC-BI-AI-96-01, 1996.

[2] J. F. Allen, "Maintaining knowledge about temporal intervals," *Commun. ACM*, vol. 26, no. 11, pp. 832–843, 1983.

[3] J. van Benthem, *The Logic of Time*. Kluwer Academic Publishers, 1991.

[4] I. Niles and A. Pease, "Towards a standard upper ontology," in *Formal Ontologies and Information Systems*, 2001, pp. 2–9.

[5] M. Fernandez-Lopez and A. Gomez-Perez, "Searching for a time ontology for semantic web applications," in *Formal Ontologies and Information Systems*, 2004, pp. 331–341.

[6] J. Hobbs and F. Pan, "Time ontology in OWL," http://www.w3.org/TR/owl-time, Tech. Rep. W3C Working Draft 27 September 2006, 2006.

[7] ——, "An ontology of time for the semantic web," *ACM Transactions on Asian Language Processing (TALIP): Special issue on Temporal Information Processing*, vol. 3, pp. 66–85, 2004.

[8] R. Fikes and Q. Zhou, "A reusable time ontology," in *Proceedings of the AAAI Workshop on Ontologies for the Semantic Web*, 2002.

[9] H. Enderton, *Mathematical Introduction to Logic*. Academic Press, 1972.

[10] M. Gruninger, T. Hahmann, A. Hashemi, and D. Ong, "Ontology verification with repositories," in *Formal Ontologies and Information Systems*, 2010, pp. 317–330.

[11] F. Buekenhout, *Handbook of Incidence Geometry*. Elsevier, 1995.

[12] L. Vila, "Formal theories of time and temporal incidence," in *Handbook of Temporal Reasoning in Artificial Intelligence*, 2005, pp. 1–24.